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## BINARY BOOSTS

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### Abstract

The relativistic motion of an isolated two-body system (bound or unbound) of given lab energy  $K^0$  in QED is separated into *cms* motion and relative motion. The relative motion equation  $\mathcal{K}_L \psi_L(\mathbf{r}_L) = 0$  contains the momentum eigenvalue  $\mathbf{K}$  of the *cms* motion. It is greatly simplified by a binary boost to the atomic rest frame, where  $K^0$  and  $\mathbf{K}$  appear only in a Lorentz-invariant combination. This boost is not a product of single-particle boosts, which are useful only for perturbative interactions. CPT-invariance is demonstrated, and orthogonality relations are derived.

### I. INTRODUCTION

This paper is about the relativistic two-body problem in QED. It treats the separation of the *cms* motion from the relative motion in analogy with the nonrelativistic separation. The case of two spinless particles has been treated previously [1] but has no practical applications. When one of the two particles is a lepton (e,  $\mu$  or  $\tau$ ) and the other is spinless, the two-body wave function contains an additional boost:

$$A = (\gamma + \gamma_5 \mathbf{K} \boldsymbol{\sigma}_1 / E)^{1/2}, \quad \gamma = K^0 / E, \quad K^{02} - \mathbf{K}^2 = E^2 \quad (1)$$

in units  $\hbar = c = 1$ . Here  $K^\mu = (K^0, \mathbf{K})$  are the eigenvalues of  $P^\mu = (i\partial_t, \mathbf{P})$  with  $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ ;  $\boldsymbol{\sigma}_1$  are the Pauli matrices of particle 1, which are normally combined with the Dirac matrix  $\gamma_5$  into  $\boldsymbol{\alpha}_1 = \gamma_5 \boldsymbol{\sigma}_1$ . The *binary boost*  $A$  resembles the boost  $A_1$  for a single free particle:

$$A_1 = (\gamma_1 + \gamma_5 \mathbf{K}_1 \boldsymbol{\sigma}_1 / m_1)^{1/2}, \quad \gamma_1 = K_1^0 / m_1, \quad K_1^{02} - \mathbf{K}_1^2 = m_1^2. \quad (2)$$

(An alternative form of  $A_1$  which avoids  $\boldsymbol{\sigma}_1$  under the square root is  $(2m_1)^{-1/2} (K_1^0 + m_1)^{-1/2} (K_1^0 + m_1 + \boldsymbol{\alpha}_1 \mathbf{K}_1)$ , but (2) is in fact more elegant.) These points will be elaborated in section III. They are of interest for muonic helium ( $\mu^- \alpha$ ), mu-pionium ( $\mu^- \pi^+$ ) and also for precise recoil corrections in ordinary atoms.

Binary boosts are simplified by taking the  $z$ -axis along the total momentum  $\mathbf{K}$ ,  $\sigma_1 \mathbf{K} = \sigma_{1z} K$ , and by abbreviating  $K/E = \gamma v/c$  as  $\widehat{K}$ :  $A = (\gamma + \gamma_5 \widehat{K} \sigma_{1z})^{1/2}$ . The symbol  $\beta$  will denote the *parity Dirac matrix* which will be taken in diagonal form:

$$\beta = \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3)$$

The boost which we find for two spinor particles contains  $\sigma_z = \sigma_{1z} + \sigma_{2z}$  as well as a  $\beta$ -dependent mass ratio  $\mu_\beta$ :

$$B = \left(1 + \frac{1}{2} \widehat{K}^2 \sigma_z^2 + \mu_\beta \widehat{K} \sigma_z \gamma_5\right)^{1/2}, \quad \widehat{K} = K/E, \quad (4)$$

$$\mu_\beta = M_-/M_+, \quad M_\pm = m_2 \pm \beta m_1. \quad (5)$$

As  $\sigma_z^2$  has only the two eigenvalues 4 and 0, an alternative form of  $B$  is

$$B = \tilde{\gamma} + \mu_\beta \widehat{K} \sigma_z \gamma_5 / 2, \quad \tilde{\gamma} = \sqrt{1 + \widehat{K}^2 \sigma_z^2 / 4}. \quad (6)$$

Evidently,  $\tilde{\gamma} = \gamma$  for  $\sigma_z^2 = 4$  and  $\tilde{\gamma} = 1$  for  $\sigma_z^2 = 0$ .  $B$  contains two Pauli matrices but only one  $\beta$  and one  $\gamma_5$ , i.e.  $B$  is an  $8 \times 8$  matrix. For comparison, the two-body boost  $B_{12}$  used so far in perturbative QED is a  $16 \times 16$ -matrix,

$$B_{12} = A_1 A_2 = (\gamma_1 + \mathbf{p}_1 \boldsymbol{\alpha}_1 / m_1)^{1/2} (\gamma_2 + \mathbf{p}_2 \boldsymbol{\alpha}_2 / m_2)^{1/2}. \quad (7)$$

It requires two separate conserved energies,  $K_1^0$  and  $K_2^0$ , momentum operators  $\mathbf{p}_1$  and  $\mathbf{p}_2$  and Dirac matrices  $\beta_1, \beta_2, \gamma_{51}, \gamma_{52}$ . The main advantage of  $B$  is that it contains only the total momentum  $\mathbf{K}$  instead of the individual momenta. It Lorentz transforms the asymptotic part of the *cms* spinorial wave function, the radial part of which is  $e^{ikr}$  in spherical coordinates (bound states have imaginary  $k = i\kappa$ ; in QED, the ground state has  $\kappa_0 = a_B^{-1}$ ,  $a_B$  being the relativistic Bohr radius). The boost  $B_{12}$ , on the other hand, requires asymptotic wave functions  $e^{i\mathbf{k}_1 \mathbf{r}_1} e^{i\mathbf{k}_2 \mathbf{r}_2}$  and then by analogy also two different times,  $e^{-iK_1^0 t_1} e^{-iK_2^0 t_2}$ . The derivation of  $B$  is given in the next section. As in the nonrelativistic case, the shape of the potential  $V(\mathbf{r}_L)$  is irrelevant in this derivation ( $\mathbf{r}_L = \mathbf{r}_1 - \mathbf{r}_2$ ). Anomalous magnetic moments can be included in the *cms* equation [2] but are as yet excluded from the boost. Thus the present treatment applies to the bound states of two charged leptons, which may be called leptonium (muonium  $e^- \mu^+$ , antimuonium  $e^+ \mu^-$ , positronium  $e^- e^+$ ), as well as to their scattering states up to  $K^0 \rightarrow \infty$  and/or  $E \rightarrow \infty$ .

The relativistic separation of the *cms* coordinate  $\mathbf{R}$  reads [1]

$$\mathbf{r}_1 = \mathbf{R} + \frac{1}{2} \left(1 - \Delta m^2 / E^2\right) \mathbf{r}_L, \quad \mathbf{r}_2 = \mathbf{R} - \frac{1}{2} \left(1 + \Delta m^2 / E^2\right) \mathbf{r}_L \quad (8)$$

with  $\Delta m^2 = m_1^2 - m_2^2 = -M_+ M_-$ . For loosely bound states,  $E^2 \approx (m_1 + m_2)^2$  reduces (8) to the nonrelativistic transformation except for the index L in  $\mathbf{r}_L$  which still signifies the lab system. This additional index is necessary because  $\mathbf{r}_L$  is Lorentz-contracted in the direction  $\mathbf{K}$ . The uncontracted *cms* variable will be denoted by  $\mathbf{r}^*$ . In the following, the  $z$ -axis is taken along  $\mathbf{K}$ , such that  $x_L$  and  $y_L$  are not Lorentz-contracted, and

$$z_L = z^*/\gamma, \quad p_{Lz} = \gamma p_z^*. \quad (9)$$

The usual *cms* potential  $V^* = -\alpha/r^*$  ( $\alpha = e^2$ ) which is abstracted from the QED Born approximation is Lorentz-contracted in the lab system. In the 16-component Dirac-Breit equation which is applied successfully in atomic theory, this contraction is not evident, but a careful analysis shows that it is represented by the Breit operators [1]. With a simple  $V(r) = -\alpha/r$ , the *cms* equation is free of Breit operators. However, this form in fact requires a third variable transformation which affects only the distance,  $r \rightarrow r^*$ . In other words, the Fourier transform of the first Born approximation of the *cms* scattering amplitude produces a potential  $V(r)$  where  $r$  is a *quasidistance* from the Dirac-Breit point of view. These procedures are up to now only approximately Lorentz-invariant and are not repeated here. We merely wish to reserve the index-free  $\mathbf{r}$  for the quasiposition, which may or may not coincide with  $\mathbf{r}^*$ .

An important application of this 8-component formalism is to positronium. The relation  $m_2 = m_1$  simplifies (8) to  $\mathbf{r}_1 = \mathbf{R} + \mathbf{r}_L/2$ ,  $\mathbf{r}_2 = \mathbf{R} - \mathbf{r}_L/2$ . In this case,  $B$  can be used in the small components ( $\beta = -1$ ,  $\mu_\beta = \infty$ ) only for  $\sigma_z^2 = 0$ , in which case the large components have  $\sigma_z^2 = 4$ . The opposite case requires a  $\gamma_5$ -transformation, which will be explained in section II.

We have found in the literature a formula which resembles (4): Replacing  $E \rightarrow m_1$  and  $\mu_\beta \rightarrow 1$ , (4) becomes identical with a boost for a single free particle of spin 1 [3].

## II. DERIVATION OF THE 8-COMPONENT EQUATION

A convenient starting point in the lab system is at present the 16-component Dirac-Breit equation

$$\left(\pi_L^0 - m_1\beta_1 - m_2\beta_2 - P_B\right)\psi_{DL} = 0, \quad \pi_L^0 = i\partial_t - V_{12} \quad (10)$$

where  $P_B$  is the Dirac-Breit momentum operator [1], and the indices D and L stand for Dirac and lab system, respectively. It contains  $\mathbf{p}_1$  and  $\mathbf{p}_2$  multiplied by Breit corrections  $1 - V_{12}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2 + \alpha_{1r}\alpha_{2r})/2$ , which contain low-energy approximations. In the future, one would like to formulate the Feynman rules for the fermion-fermion scattering amplitudes in terms of 8-component spinors, both in the lab system and in the *cms*, and then derive the interactions via Fourier transformations. At present, this derivation is incomplete even in the *cms*. The *cms* equation has been abstracted from (10), but most of its scattering amplitudes have been checked against the Born approximation for arbitrary energies. For the present construction of the free two-particle boost, we only need three properties of (10): (i) it is translationally invariant in space and time,

$$i\partial_t\psi_{DL} = K^0\psi_{DL}, \quad (\mathbf{p}_1 + \mathbf{p}_2)\psi_{DL} = \mathbf{K}\psi_{DL}, \quad (11)$$

(ii) it is exact at all energies in the asymptotic region  $V_{12} = 0$  and (iii) it contains only a single time  $t$ , which contributes a factor  $e^{-iK^0t}$  for stationary states. The resulting time-independent Dirac-Breit equation is again (10), but with  $\pi_L^0 = K^0 - V_{12}$ . It contains the six variables  $\mathbf{r}_1$  and  $\mathbf{r}_2$ , precisely as in the nonrelativistic case. For  $V_{12} = 0$ , we thus have

$$\left(K^0 - m_1\beta_1 - m_2\beta_2 - \mathbf{p}_1\boldsymbol{\alpha}_1 - \mathbf{p}_2\boldsymbol{\alpha}_2\right)\psi_{DL} = 0. \quad (12)$$

In the parity basis in which  $\beta_1$  and  $\beta_2$  are diagonal, the components of  $\psi_{\text{DL}}$  may be labelled by the index  $g$  for a large component and  $f$  for a small component of either particle:  $\psi_{gg}$ ,  $\psi_{gf}$ ,  $\psi_{fg}$  and  $\psi_{ff}$ . In the following, these components are rearranged into sums and differences as follows:

$$\psi_{\text{DL}} = \begin{pmatrix} \psi_g \\ \psi_f \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{gg} + \psi_{ff} \\ \psi_{gf} + \psi_{fg} \end{pmatrix}, \quad \chi_{\text{DL}} = \begin{pmatrix} \chi_g \\ \chi_f \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{gg} - \psi_{ff} \\ \psi_{gf} - \psi_{fg} \end{pmatrix}. \quad (13)$$

The index D stands for Dirac, the index L for the lab system. The factor  $2^{-1/2}$  makes the transformation unitary. The 16-component equation thus assumes the form of two coupled 8-component equations,

$$(K^0 - \gamma_5 p_+) \psi_{\text{DL}} = \beta M_+ \chi_{\text{DL}}, \quad (K^0 + \gamma_5 p_-) \chi_{\text{DL}} = \beta M_+ \psi_{\text{DL}}, \quad (14)$$

$$p_{\pm} = \mathbf{p}_1 \boldsymbol{\sigma}_1 \pm \mathbf{p}_2 \boldsymbol{\sigma}_2, \quad (15)$$

with  $\beta, \gamma_5$  and  $M_+$  defined in (3) and (5). In words,  $\gamma_5$  exchanges the single index  $g$  with the single index  $f$  in  $\psi$  and  $\chi$ , while  $\beta$  multiplies  $f$  by  $-1$ . The original matrices  $\beta_1, \beta_2, \gamma_{51}$  and  $\gamma_{52}$  are now obsolete. Elimination of  $\chi_{\text{DL}}$  by means of the first equation (14) yields

$$(K^0 + \gamma_5 p_-) \beta M_+^{-1} (K^0 - \gamma_5 p_+) \psi_{\text{DL}} = \beta M_+ \psi_{\text{DL}}. \quad (16)$$

Upon multiplication by  $\beta M_+$ , this becomes

$$(K^{02} - M_+^2 - K^0 p_- \mu_{\beta}^{-1} \gamma_5 - K^0 p_+ \gamma_5 + \mu_{\beta}^{-1} p_- p_+) \psi_{\text{DL}} = 0. \quad (17)$$

The denominator  $M_- = m_2 - \beta m_1$  appears here because of  $\gamma_5 \beta = -\beta \gamma_5$ . At this point, we replace  $\mathbf{r}_1$  and  $\mathbf{r}_2$  by  $\mathbf{R}$  and  $\mathbf{r}_L$  according to (8), which leads to

$$\mathbf{p}_1 = \mathbf{p}_L + \frac{1}{2} (1 + \Delta m^2 / E^2) \mathbf{K}, \quad \mathbf{p}_2 = -\mathbf{p}_L + \frac{1}{2} (1 - \Delta m^2 / E^2) \mathbf{K} \quad (18)$$

where (11) has already been used, and  $E^2 = K^{02} - \mathbf{K}^2$  as in (1).  $\psi_{\text{DL}}$  is now a function of the vector  $\mathbf{r}_L$  and of the parameters  $K^0$  and  $K$  (with the  $z$ -axis along  $\mathbf{K}$ ). We now show that  $\psi_{\text{DL}}$  can be reduced to a function of the components  $\mathbf{r}_{Lt} = (x_L, y_L)$  and  $z^* = \gamma z_L$  which contains  $K^0$  and  $K$  only in the combination  $K^{02} - K^2 = E^2$ . This justifies the choice of the transformation (8), (9) a posteriori.

The combinations  $p_{\pm}$  of (15) become

$$\begin{aligned} p_- &= \mathbf{p}_L \boldsymbol{\sigma}_t + \left( \gamma p_z^* + \frac{\Delta m^2}{2E^2} K \right) \sigma_z + \frac{1}{2} K \Delta \sigma_z, \\ p_+ &= \mathbf{p}_L \Delta \boldsymbol{\sigma}_t + \left( \gamma p_z^* + \frac{\Delta m^2}{2E^2} K \right) \Delta \sigma_z + \frac{1}{2} K \sigma_z, \end{aligned} \quad (19)$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2, \quad \Delta \boldsymbol{\sigma} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \quad \boldsymbol{\sigma}_t = (\sigma_x, \sigma_y). \quad (20)$$

Here we have denoted  $\mathbf{p}_{Lt} \boldsymbol{\sigma}_t = \mathbf{p}_L \boldsymbol{\sigma}_t$  in order to save one index. The operators  $\mathbf{p}_{Lt}$  and  $p_z^*$  refer to the *cms*, but we avoid the notation  $\mathbf{p} = -i\nabla$  which is reserved for the quasimomentum.

In the Pauli spinor space  $\chi_1 \chi_2$ ,  $\boldsymbol{\sigma}$  is symmetric in 1 and 2. It transforms spin triplets into triplets and annihilates spin singlets.  $\Delta\boldsymbol{\sigma}$  is antisymmetric and exchanges singlets ( $\psi_{Ls}$ ) with triplets ( $\psi_{Lt}$ ). It turns out that the following separation of mass factors is useful:

$$\psi_{DL} = \begin{pmatrix} \psi_{DLt} \\ \psi_{DLs} \end{pmatrix} = \begin{pmatrix} M_+ \psi_{Lt} \\ M_- \psi_{Ls} \end{pmatrix}. \quad (21)$$

The index D disappears here; the new components  $\psi_{Lt}$  and  $\psi_{Ls}$  form an 8-component spinor  $\psi_L$ , which satisfies the following equation:

$$\left\{ K^{02} - M_+^2 + \mu_\beta^{-1} p_- p_+ - K^0 \left[ 2\mathbf{p}_L \boldsymbol{\sigma}_{1t} + \left( 2\gamma p_z^* + \frac{\Delta m^2 K}{E^2} \right) \sigma_{1z} + \frac{K}{2} \left( \mu_\beta^{-1} \Delta \sigma_z + \mu_\beta \sigma_z \right) \right] \gamma_5 \right\} \psi_L = 0. \quad (22)$$

Here we have used  $\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma} = 2\boldsymbol{\sigma}_1$ . The expression  $p_- p_+$  simplifies considerably,

$$p_- p_+ = \frac{\Delta m^2}{E^2} K^2 + 2K\gamma p_z^*. \quad (23)$$

Remembering  $\Delta m^2 = -M_+ M_-$ , the form  $M_+ \Delta m^2 K^2 / E^2 M_- = -M_+^2 K^2 / E^2$  is combined with  $-M_+^2$  into  $-M_+^2 (1 + K^2 / E^2) = -M_+^2 \gamma^2$ ,  $\gamma = K^0 / E$ . One may thus remove one factor  $\gamma$  from (22). The result is  $\mathcal{K}_L \psi_L = 0$ , where

$$\mathcal{K}_L = (E^2 - M_+^2) \gamma + 2K \mu_\beta^{-1} p_z^* - \left[ 2E \mathbf{p}_L \boldsymbol{\sigma}_{1t} + (2E \gamma p_z^* - M_+ M_- \widehat{K}) \sigma_{1z} + \frac{1}{2} \widehat{K} E^2 \left( \mu_\beta^{-1} \Delta \sigma_z + \mu_\beta \sigma_z \right) \right] \gamma_5, \quad (24)$$

with  $\widehat{K} = K/E$  as before. In the *cms*, i.e.  $\widehat{K} = 0$ ,  $\gamma = 1$ , this reduces to

$$\mathcal{K} \psi = 0, \quad \mathcal{K} = E^2 - M_+^2 - 2E(\mathbf{p}_L \boldsymbol{\sigma}_{1t} + p_z^* \sigma_{1z}) \gamma_5. \quad (25)$$

The operator  $\mathcal{K}$  is independent of  $\boldsymbol{\sigma}_2$  and well-behaved for  $m_1 = m_2$ , where  $M_+^2 = 2m_1^2(1+\beta)$  annihilates the small components. It remains to find a transformation  $\psi_L = B\psi$  which reduces  $\mathcal{K}_L$  to  $\mathcal{K}$  after multiplication by another matrix  $\bar{B}$  from the left

$$\bar{B} \mathcal{K}_L B \psi = \mathcal{K} \psi = 0, \quad \mathcal{K} = \bar{B} \mathcal{K}_L B. \quad (26)$$

As the operator  $-2E \mathbf{p}_L \boldsymbol{\sigma}_{1t}$  occurs in (25) in the same form as in (24), one needs

$$\bar{B} \boldsymbol{\sigma}_{1t} = \boldsymbol{\sigma}_{1t} B^{-1}. \quad (27)$$

The form (24) suggests that  $B$  may not contain  $\boldsymbol{\sigma}_{1t}$  and  $\boldsymbol{\sigma}_{2t}$ . Next, one may note that

$$\sigma_{1z} \boldsymbol{\sigma}_{1t} = -\boldsymbol{\sigma}_{1t} \sigma_{1z}, \quad \Delta \sigma_z \boldsymbol{\sigma}_{1t} = -\boldsymbol{\sigma}_{1t} \sigma_z. \quad (28)$$

After a few fruitless attempts, one solves (26) with

$$B = \left(1 + \frac{1}{2}\widehat{K}^2\sigma_z^2 + \mu_\beta\widehat{K}\sigma_z\gamma\gamma_5\right)^{1/2}, \quad B^{-1} = \left(1 + \frac{1}{2}\widehat{K}^2\sigma_z^2 - \mu_\beta\widehat{K}\sigma_z\gamma\gamma_5\right)^{1/2}, \quad (29)$$

$$\bar{B} = \left(1 + \frac{1}{2}\widehat{K}^2\Delta\sigma_z^2 + \mu_\beta^{-1}\widehat{K}\Delta\sigma_z\gamma\gamma_5\right)^{1/2}. \quad (30)$$

It remains to show how the eigenvalue  $\infty$  of  $\mu_\beta$  (5) is avoided for positronium. When the large components have  $\sigma_z^2 = 4$ ,  $\sigma_z = \pm 2$  implies  $m_l = m_j \pm 1$  ( $m_l$  and  $m_j$  are the eigenvalues of  $L_z$  and  $J_z = L_z + \sigma_z/2$ ). These states have parity  $(-1)^{j+1}$  ( $J^2 = j(j+1)$ ). The orbital parity of the small components is opposite, i.e.  $(-1)^j$ . The spin function can be either singlet or triplet, and in either case  $m_l = m_j$  ensures  $\sigma_z = 0$ , i.e. the combination  $\mu_\beta\sigma_z$  can be taken to vanish in (6). Thus in the application to positronium, one must discuss the spin structure for  $m_1 \neq m_2$  and then take  $m_1 = m_2$  in the final forms. When the large components have  $\sigma_z = 0$ , a chiral transformation  $\psi_{\text{DL}} = \gamma_5\psi_{\text{DL},ch}$  transforms  $\mu_\beta$  into  $\mu_\beta^{-1}$ , such that the above argument applies again.

### III. THE KLEIN-DIRAC BOOST

The 4-component Dirac-Klein-Gordon-Breit equation for systems such as  $\mu^-\pi^+$  reads [1]

$$\left(\pi_{\text{L}}^{02} + \mathcal{K}_1 - \mathcal{K}_2 - 2m_1\pi_{\text{L}}^0\beta\right)\psi'_{\text{L}} = \gamma_5\left(\{\mathbf{p}_1, \pi_{\text{L}}^0\}\boldsymbol{\sigma}_1 + b_1\right)\psi'_{\text{L}} \quad (31)$$

where  $b_1$  is the Breit modification of the momentum operator,

$$b_1 = -V_{12}(\boldsymbol{\sigma}_1\mathbf{p}_2 + \sigma_{1r}p_{2r}), \quad (32)$$

and  $\mathcal{K}_i = m_i^2 + p_i^2$ . The coordinate transformations (8) and (9) give

$$\mathcal{K}_1 - \mathcal{K}_2 = \gamma^2\Delta m^2 + 2\gamma p_z^*K. \quad (33)$$

Setting now  $V_{12} = 0$  and extracting one factor  $\gamma$ , one obtains  $\mathcal{K}'_{\text{L}}\psi'_{\text{L}} = 0$ , with

$$\mathcal{K}'_{\text{L}} = \left(E^2 + \Delta m^2\right)\left(\gamma - \gamma_5\widehat{K}\sigma_{1z}\right) + 2Kp_z^* - 2\gamma_5E(\gamma p_z^*\sigma_{1z} + \mathbf{p}_{\text{L}}\boldsymbol{\sigma}_{1t}) - 2m_1\beta E. \quad (34)$$

Writing the lab spinor  $\psi_{\text{L}}$  as a boost  $A$  times the *cms* spinor  $\psi'$ ,  $\psi'_{\text{L}} = A\psi'$ , one finds

$$\mathcal{K}'\psi' = 0, \quad \mathcal{K}' = A\mathcal{K}'_{\text{L}}A, \quad A = \left(\gamma + \gamma_5\widehat{K}\sigma_{1z}\right)^{1/2}, \quad (35)$$

$$\mathcal{K}' = E^2 + \Delta m^2 - 2Em_1\beta - 2E(p_z^*\alpha_z + \mathbf{p}_{\text{L}}\boldsymbol{\alpha}_t). \quad (36)$$

The asymptotic energy  $E_1$  of the spinor particle is  $E_1 = (E^2 + \Delta m^2)/2E$ , such that the first two terms in (36) arise from  $2EE_1$ . As  $\mathcal{K}'$  has the Dirac operator structure, it is clear that its eigenvalues depend on  $E^2 + \Delta m^2$  and  $-2Em_1\beta$  only via  $(E^2 + \Delta m^2)^2 - 4m_1^2E^2$ . Setting now

$$\psi' = \left(2k^2E\right)^{-1/2}(E_1 - \beta m_1)^{1/2}\left(E^2 - M^2\right)^{1/2}\psi, \quad k^2 = E_1^2 - m_1^2 \quad (37)$$

one finds that  $\mathcal{K}'$  is transformed into  $\mathcal{K}$  as given in (25) (the resulting boost is rather different, however). The point  $k^2 = 0$  comprises two thresholds at  $E^2 = (m_1 + m_2)^2$  and two at  $E^2 = (m_1 - m_2)^2$ . In our example of a  $\mu^- \pi^+$  system,  $E = m_1 + m_2$  is the  $\mu^- \pi^+$  threshold,  $E = -m_1 - m_2$  the  $\mu^+ \pi^-$  threshold,  $E = m_2 - m_1$  the  $\mu^+ \pi^+$  threshold and  $E = m_1 - m_2$  the  $\mu^- \pi^-$  threshold. For  $m_1 = m_2$ , one factor  $E$  can be separated from (36), leading to  $\mathcal{K}' = E - 2m_1\beta - 2(p_z^* \alpha_z + \mathbf{p}_L \boldsymbol{\alpha}_t)$ . This completely eliminates the doubly-charged channels, which now require a separate equation. A similar decoupling occurs in the double-Dirac case for  $m_1 = m_2$ , which is discussed below.

#### IV. ORTHOGONALITY RELATIONS

A single 4-component Dirac spinor has components  $\psi_R$  and  $\psi_L$  in the chiral basis, where  $\gamma_5$  is diagonal.  $\beta$  and  $\gamma_5$  simply change their places in (3), i.e.  $\beta$  exchanges  $\psi_R$  and  $\psi_L$ . As  $\beta$  is part of the parity transformation,  $\psi_R$  and  $\psi_L$  are not parity eigenstates. They do form separate representations of the Lorentz group, however.

In the present case, both  $\psi$  and  $\chi$  can have  $\beta$  diagonal and remain separate under Lorentz transformations. Elimination of  $\psi_{DL}$  from (14) leads to an equation for  $\chi_{DL}$  in which the two brackets of (16) are interchanged, which is equivalent to the substitution  $p_+ \longleftrightarrow -p_-$ . The equation corresponding to (17) for  $\chi_{DL}$  is thus

$$\left(K^2 - M_+^2 + (\mu_\beta^{-1} p_+ + p_-) K^0 \gamma_5 + \mu_\beta^{-1} p_+ p_-\right) \chi_{DL} = 0. \quad (38)$$

It will be shown in the following that the orthogonality relations require both  $\psi$  and  $\chi$ . The mass separation analogous to (21) is

$$\chi_{DL} = \beta \begin{pmatrix} \chi_{Lt} \\ \mu_\beta^{-1} \chi_{Ls} \end{pmatrix}. \quad (39)$$

The relation analogous to (24) is  $\mathcal{K}_{\chi L} \chi_L = 0$ , with

$$\begin{aligned} \mathcal{K}_{\chi L} = (E^2 - M_+^2) \gamma + 2K \mu_\beta^{-1} p_z^* - \left[ 2E \mathbf{p}_L \boldsymbol{\sigma}_{1t} + (2E \gamma p_z^* - M_+ M_- \widehat{K}) \sigma_{1z} + \right. \\ \left. + \frac{1}{2} \widehat{K} E^2 (\mu_\beta \Delta \sigma_z + \mu_\beta^{-1} \sigma_z) \right] \gamma_5, \end{aligned} \quad (40)$$

The boost is now different,  $\chi_L = B_\chi \chi$ , where

$$B_\chi = \left( 1 + \frac{1}{2} \widehat{K}^2 \Delta \sigma_z^2 + \mu_\beta \widehat{K} \Delta \sigma_z \gamma \gamma_5 \right)^{1/2} = \bar{B}^\dagger \quad (41)$$

where  $\bar{B}$  has been given in (30). The hermitian conjugation of  $\bar{B}$  in (30) merely exchanges the positions of  $\mu_\beta^{-1}$  and  $\gamma_5$ . Observing  $(\mu_\beta^{-1} \gamma_5)^\dagger = \gamma_5 \mu_\beta^{-1} = \mu_\beta \gamma_5$ , one readily verifies (41). The coordinate transformation (8) implies  $d^3 r_1 d^3 r_2 = d^3 r_L d^3 R$ , and the orthogonality of states with different momenta  $\mathbf{K}$  follows simply from  $\int d^3 R \exp\{-i(\mathbf{K} - \mathbf{K}')\mathbf{R}\} = (2\pi)^3 \delta(\mathbf{K} - \mathbf{K}')$ . A subsequent boost allows one to discuss the orthogonality of the functions  $\psi(\mathbf{r}_L)$  in the *cms*. The relevant equations here are  $\mathcal{K}\psi = 0$  and  $\mathcal{K}\chi = 0$ , with  $\mathcal{K}$  given by (25). The latter equation follows from  $\mathcal{K}_{\chi L} \chi_L = 0$  for  $\widehat{K} = 0$ ,  $\gamma = 1$ , i.e.  $\mathcal{K}_\chi = \mathcal{K}$  in the *cms*.

In the ordinary Dirac equation, neither the potential  $V(r)$  nor the momentum  $\mathbf{p} = -i\nabla$  appear in the orthogonality relations. Both properties are also achieved here, but the details are surprising. The general operator form of  $\mathcal{K}$  will be needed for all values of  $r$ , not just for  $r \rightarrow \infty$  where  $V(r) = 0$ . This form can be derived from the Fourier transform of the QED Born approximation for elastic scattering  $1 + 2 \rightarrow 1' + 2'$  in the *cms*:

$$\mathcal{K} = E^2 - M_+^2 - 2E\boldsymbol{\alpha}\mathbf{p} - 2EV(r) - \Lambda V'(r), \quad (42)$$

$$\boldsymbol{\alpha} = \gamma_5 \boldsymbol{\sigma}_1, \quad \Lambda = i\alpha_r \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 = (\boldsymbol{\alpha} \times \boldsymbol{\sigma}_2)_r + i\gamma_5 \sigma_{2r}, \quad (43)$$

and  $V' = dV/dr$ ,  $\sigma_r = \boldsymbol{\sigma} \hat{\mathbf{r}}$  as usual. The derivation from the Dirac–Breit equation leads to a more complicated form in the variable  $\mathbf{r}^* = (z^*, x_L, y_L)$ , which involves Breit operators and also  $V^2(r^*)$ . However, it turns out that the substitution

$$r^* \approx r + \alpha/2E \quad (44)$$

does reduce the Dirac–Breit expression to (42) at low energies. As the QED Born approximation is free of low-energy approximations, (42) is much more practical than the Dirac–Breit equation.

To derive the orthogonality relations for the solution of a Hamiltonian equation  $H\psi = E\psi$ , one writes  $H\psi_j = E_j\psi_j$ ,  $(H\psi_i)^\dagger = E_i\psi_i^\dagger$ , multiplies the first equation by  $\psi_i^\dagger$ , the second one by  $\psi_j$ , subtracts the second product from the first one, and integrates over all configuration space:  $(E_i - E_j) \int \psi_i^\dagger \psi_j = 0$ . The method can also be applied to the Klein–Gordon equation,  $[(E - V(r))^2 + \nabla^2 - m^2]\psi = 0$  (in units  $\hbar = c = 1$ ), but the square of  $E - V$  produces a weight  $w_{ij} = E_i + E_j - 2V$ , i.e.  $\int \psi_i^* w_{ij} \psi_j = \delta_{ij}$ . The case at hand is more complicated because  $\Lambda$  is not hermitian. Its hermitian part is the recoil-corrected hyperfine operator. Its antihermitian part has zero expectation values in fine structure eigenstates; it is needed in positronium where fine and hyperfine structures are comparable (the energy eigenvalues remain real). Fortunately, the equation  $\mathcal{K}_\chi \chi = 0$  has  $\Lambda$  replaced by  $\Lambda^\dagger$ :

$$\mathcal{K}_\chi = E^2 - M_+^2 - 2E\boldsymbol{\alpha}\mathbf{p} - 2EV(r) - \Lambda^\dagger V'(r). \quad (45)$$

One may thus envisage orthogonality relations  $\int \chi_i^\dagger w_{ij} \chi_j = \delta_{ij}$ , where the antihermitian component disappears with  $\Lambda^{\dagger\dagger} - \Lambda = 0$ . However, the factor  $E$  multiplies not only  $V(r)$  as in the Klein–Gordon equation, but also  $\boldsymbol{\alpha}\mathbf{p}$ . To avoid  $\mathbf{p}$  in the orthogonality relations,  $E$  must be divided off. But then  $\Lambda/E$  is both non-hermitian and energy-dependent, in which case  $V'$  will remain in the orthogonality relations.

For  $V = -\alpha/r$ , one may introduce a dimensionless scaled variable,

$$\tilde{r} = Er \quad \partial/\partial\tilde{r} = E^{-1}\partial/\partial r \quad \tilde{\mathbf{p}} = \mathbf{p}/E, \quad (46)$$

and divide  $\mathcal{K}$  by  $E^2 = s$ :

$$(2\boldsymbol{\alpha}\tilde{\mathbf{p}} + 2V(\tilde{r}) + \Lambda V'(\tilde{r}) - 1 + M_+^2/s) \psi(\tilde{r}) = 0. \quad (47)$$

Using the corresponding equation for  $\chi^\dagger$ , one arrives at



$$(s_i^{-1} - s_j^{-1}) \int \chi_i^\dagger M_+^2 \psi_j d^3\tilde{r} = 0, \quad s_i = E_i^2, \quad (48)$$

$$\int \chi_i^\dagger M_+^2 \psi_j d^3\tilde{r} = \delta_{ij}. \quad (49)$$

Remembering  $M_+^2 = m_1^2 + m_2^2 + 2m_1m_2\beta$ , this is a simple generalization of the static limit  $m_1/m_2 = 0$ . For positronium, the small components do not contribute to (49).

Equation (48) is explicitly CPT-invariant: Every bound state  $s_i$  has two different eigenvalues  $E_i$ , namely  $E_i = \sqrt{s_i} \equiv m_{Ai}$  and  $E_i = -\sqrt{s_i} \equiv -m_{Ai}$ , where  $m_{Ai}$  denotes the atomic mass in the state  $i$  (an excited atom is heavier than its ground state). Returning now to the time-dependent *cms* equation with  $i\partial_t = E$ , one finds that  $E_i = -m_{Ai}$  belongs to the antiatom of mass  $m_{\bar{A}i}$ , i.e.  $m_{\bar{A}i} = m_{Ai}$ . Positronium is its own antiatom, of course.

Vacuum polarization introduces an extra scale into  $V(r)$ . The equation remains CPT-invariant, but (48) becomes more complicated. In the Klein-Dirac case of section III, the spinless particle normally has an extended charge distribution, leading to  $V(r) \neq -\alpha/r$  for small  $r$ . The simplest way out is of course to take  $V(\tilde{r})$  as an arbitrary function of  $\tilde{r}$  in (47). In any case, finite charge distribution models need some tuning in order to conform with CPT.

It should be stated that the range of the dimensionless radial variable  $\tilde{r}$  is  $0 < \tilde{r} < \infty$  both for atoms and for antiatoms. In the old variable  $r$ , antiatoms have *negative distances*. Of course, this makes as little sense as the claim that antiatoms *fly backwards in time*.

We conclude with two comments on possible applications of the new equations: A vector potential  $\mathbf{A}(\mathbf{r}, t)$  is included in (12) by replacing  $\mathbf{p}_i \rightarrow \boldsymbol{\pi}_i = \mathbf{p}_i + q_i \mathbf{A}(\mathbf{r}_i, t)$ . This allows one to calculate relativistic recoil effects in positronium de-excitation, or in the  $e^+e^-$ -recombination into positronium. In addition, the Lamb shift may be calculated from these processes via a dispersion integral.

The second comment concerns the case  $m_1 = m_2$ . Here  $M_+^2/s$  vanishes for the small components  $\psi_f$  of  $\psi$ . Calling for brevity

$$\tilde{V} = V(\tilde{r}), \quad \tilde{\boldsymbol{\pi}} = \tilde{\mathbf{p}} + \frac{i}{2} \tilde{\nabla} \tilde{V} \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2, \quad (50)$$

one has  $\psi_f = (1 - 2\tilde{V})^{-1} 2\boldsymbol{\sigma}_1 \tilde{\boldsymbol{\pi}} \psi_g$ . The large components satisfy the equation

$$\left[ 4m_1^2/s - 1 + 2\tilde{V} + 4\boldsymbol{\sigma}_1 \tilde{\boldsymbol{\pi}} (1 - 2\tilde{V})^{-1} \boldsymbol{\sigma}_1 \tilde{\boldsymbol{\pi}} \right] \psi_g = 0. \quad (51)$$

For  $\tilde{V} = +\alpha/\tilde{r}$ , it has only scattering states, which would be  $e^-e^-$  and  $e^+e^+$  in the case of two electrons. If it were possible to extend the present formalism to an external Coulomb potential, one would arrive at a theory which isolates the  $e^-e^-$  and  $e^+e^+$ -channels from the  $e^-e^+$ -channel. This would dispense with *positive-energy projectors* and simplify the relativistic variational calculation of atomic ground states.

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## REFERENCES

- [1] M. Malveti and H. Pilkuhn, Phys.Reports **248**, 1 (1994), and references therein
- [2] H. Pilkuhn, J.Phys. B **28**, 4421 (1995)
- [3] S. Weinberg, Phys.Rev. **133**, B1318 (1964)